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Periodic orbit action correlations in the Baker map

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Abstract. Periodic orbit action correlations are studied for the piecewise linear, area-preserving Baker map. Semiclassical periodic orbit formulae together with universal spectral statistics in the corresponding quantum Baker map suggest the existence of universal periodic orbit correlations. The calculation of periodic orbit sums for the Baker map can be performed with the help of a Perron–Frobenius-type operator. This makes it possible to study periodic orbit correlations for orbits with period up to 500 iterations of the map. Periodic orbit correlations are found to agree quantitatively with the predictions from random matrix theory up to a critical length determined by the semiclassical error. Exponentially increasing terms dominate the correlations for longer orbits which are due to the violation of unitarity in the semiclassical approximation.

1. Introduction

Numerical evidence suggests that eigenvalue spectra of individual low dimensional quantum systems show correlations which are solely determined by the underlying classical dynamics, symmetries and Planck's constant $\hbar$ (see e.g. Bohigas *et al* 1984, Berry 1987, Bohigas 1991). Especially quantum systems exhibiting fully chaotic or completely regular dynamics in the classical limit show spectral fluctuations coinciding with random matrix theory (RMT). Semiclassical trace formulae provide a link between the set of eigenenergies and the set of all periodic orbits in the classical system and have proven to be important in understanding universality in spectral statistics and deviations thereof (Hannay and Ozorio de Almeida 1984, Berry 1985, Berry 1988, Aurich and Steiner 1995, Bogomolny and Keating 1996b). A complete description of spectral properties in terms of the underlying classical dynamics of the system is, however, still missing.

Universal spectral statistics is intimately connected to correlations in periodic orbit length or actions distributions. The validity of the random matrix conjecture implies generalized universal periodic action correlation for systems for which the trace formula is exact (Argaman *et al* 1993). Keating (1993) applied this idea to the Riemann ζ-function whose non-trivial zeros are conjectured to follow random matrix statistics. Here, the Hardy–Littlewood conjecture provides an explicit expression for the pair-correlations of prime numbers, (the analogue of periodic orbits in the Riemann case). Expressions for the two-point correlation function of Riemann zeros can be obtained making use of the conjectured prime number correlations

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which coincide asymptotically with RMT results for an ensemble of Gaussian random matrices invariant under unitary transformation (GUE). A generalization of these arguments to n-point correlation functions for Riemann-zeros has been given by Bogomolny and Keating (1995, 1996a).

Far less is known about action correlations of periodic orbits for generic chaotic systems and even numerical evidence for the existence of these universal periodic orbit correlations is weak. Argaman *et al* (1993) present numerical results which agree qualitatively with the RMT prediction presented in the same paper; Cohen *et al* (1998) find correlations in the length spectrum of periodic orbits in the stadium billiard which indicate RMT-like correlations but a quantitative analysis cannot compete with the accuracy reached for spectral eigenvalue statistics. Other groups report complete failure in their search for any kind of action correlations (Aurich 1998) or find action correlations which do not coincide with the RMT prediction by Argaman *et al* (1993), see Sano (1999); in addition there are yet no arguments other than from the duality between quantum eigenvalues and classical periodic orbits which indicate why universal classical correlations should exist. Cohen *et al* (1998) present a variety of ideas based on identifying relationships within various subgroups of orbits in the Sinai billiard. The results are, however, still of speculative nature.

In this paper I will study quantum and semiclassical spectra and discuss necessary conditions for the existence of universal periodic orbit action correlations. I will focus in particular on the classical and quantum Baker map. It will become evident that the periodic orbit correlations deviate from the universal behaviour for long orbits due to the violation of quantum unitarity in the semiclassical approximations. Correlations following the RMT-prediction do, however, exist for short periodic orbits which cannot be explained by classical sum rules alone. Properties of periodic orbits can be studied in detail for the Baker map by writing periodic orbit sums in terms of a suitable classical Perron–Frobenius operator. Exponentially increasing terms in the large action limit can be regularized by imposing unitarity onto a semiclassical quantization as proposed by Bogomolny and Keating (1996b).

2. Periodic orbit correlation functions for quantum maps

In the following I will limit the discussion to quantum maps. A quantum map acts on a finite dimensional Hilbert space and the quantum dynamics is governed by the equation

$$\psi_{n+1} = \boldsymbol{U}\psi_n \tag{1}$$

with $\boldsymbol{U}$ being a unitary matrix of dimension N. We assume that the map has a well-defined classical limit for $N \to \infty$ described by a discrete dynamical map. ψ_n is the discretized N-dimensional wave vector and N is equivalent to the inverse of Planck's constant h, i.e. $N \sim 1/h$. For possible quantization procedures of classical maps see, e.g., Hannay and Berry (1980) for the cat map, Balazs and Voros (1989) and Saraceno and Voros (1994) for the Baker map or Bogomolny (1992), Doron and Smilansky (1991), and Prosen (1994, 1995) for semiclassical and quantum Poincaré maps. The spectral density of eigenphases θ_i of $\boldsymbol{U}$ can be written in the form

$$d(\theta, N) = \sum_{i=1}^{N} \delta_{2\pi}(\theta - \theta_i) = \frac{N}{2\pi} + \frac{1}{\pi}\mathrm{Re}\sum_{n=1}^{\infty}\mathrm{Tr}\,\boldsymbol{U}^n \mathrm{e}^{-\mathrm{i}n\theta} \tag{2}$$

with $\overline{d} = N/2\pi$ being the mean density of eigenphases in the interval $[0, 2\pi]$ and $\delta_{2\pi}$ denotes the periodically continued δ-function with period 2π. The spectral measure of interest here is the two-point correlation function defined as

$$R_2(x,N) = \frac{1}{\overline{d}^2}\langle d(\theta)d(\theta + x/\overline{d})\rangle. \tag{3}$$

The average is taken over the unit circle, i.e. $\langle\cdot\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cdot\mathrm{d}\theta$ which leads to

$$R_2(x,N) = \frac{1}{N}\sum_{i,j=1}^{N}\delta_{2\pi}(x_i - x_j - x) = 1 + \frac{2}{N^2}\sum_{n=1}^{\infty}|\operatorname{Tr}\boldsymbol{U}^n(N)|^2\cos\left(2\pi\frac{n}{N}x\right). \tag{4}$$

The spectrum $\{x_i = \overline{d}\theta_i;\ i = 1, 2 \ldots N\}$ is unfolded to mean level density one. Note that R_2, as defined in (3), is still a distribution and further averaging has to be applied. Fourier transformation of the two-point correlation function yields the so-called form factor which can be written as

$$K(\tau,N) = \begin{cases} \frac{1}{N}|\operatorname{Tr}\boldsymbol{U}^{N\tau}(N)|^2 & \text{for} \quad \tau \neq 0 \\ N & \text{for} \quad \tau = 0 \end{cases}. \tag{5}$$

The variable τ takes the discrete values $\tau = n/N$, $n, N \in \mathbb{N}$ which reflects the periodicity of the two-point correlation function R_2. Note that the form factor does not converge in the limit $N \to \infty$ but fluctuates around a mean value. The limit distribution of $K(\tau, N)$ for $N \to \infty$ is, however, expected to exist and is conjectured to be Gaussian (Prange 1997). I will refer to the form factor as the mean value of the distribution (5) near a given τ value, i.e. further averaging over small τ-intervals has to be performed.

The traces of $\boldsymbol{U}^n$ can in a semiclassical approximation be written as sum over periodic orbits of topological length n of the underlying classical map (Gutzwiller 1990, Bogomolny 1992), i.e.

$$\operatorname{Tr}\boldsymbol{U}^n(N) \approx \operatorname{Tr}\tilde{\boldsymbol{U}}^n(N) := \sum_{p}^{(n)} A_p \mathrm{e}^{2\pi\mathrm{i}NS_p}. \tag{6}$$

The complex prefactors A_p contain information about the stability of the periodic orbit and S_p denotes the action or generating function of the map corresponding to the periodic orbit p. The relation between quantum traces and semiclassical periodic orbit sums is in general obtained by stationary phase approximation and is exact only in the limit $\tau^{-1} = N/n \to \infty$ (Andersson and Melrose 1997). Exceptions thereof are, e.g., the cat map (Hannay and Berry 1980, Keating 1991) or quantum graphs (Kottos and Smilansky 1997) for which equation (6) is exact for all τ.

Relation (6) makes it possible to connect purely classical action correlation functions with the statistics of quantum eigenvalues. A weighted action correlation function can be obtained by considering

$$P(s,n) = \sum_{N=-\infty}^{\infty}|\operatorname{Tr}\tilde{\boldsymbol{U}}^n(N)|^2\mathrm{e}^{2\pi\mathrm{i}sN} = \sum_{p,p'}^{(n)} A_p A_{p'}^*\delta_{2\pi}(S_p - S_{p'} - s) \tag{7}$$

which is a Fourier sum in N depending only on classical quantities such as the topological length of the orbit, n, and the actions S_p and weights A_p. Equations (4) and (7) together with (5) indicate that energy level statistics and periodic orbit action correlations are connected provided that both the quantum traces $\operatorname{Tr}\boldsymbol{U}^n$ and the semiclassical 'traces' $\operatorname{Tr}\tilde{\boldsymbol{U}}^n$ follow the same probability distributions, i.e. random matrix statistics. Little is known about the statistical properties of semiclassical traces and eigenvalues and I will study the existence of universal RMT-behaviour for semiclassical expressions in more detail in section 4.

The RMT conjecture implies for quantum traces $\mathrm{Tr}\,\boldsymbol{U}^n(N)$ of systems whose classical limit is chaotic and non-time reversal symmetric

$$\lim_{N\to\infty}\frac{1}{N}\langle|\,\mathrm{Tr}\,\boldsymbol{U}^n(N)|^2\rangle = \begin{cases} n/N & \text{for} \quad n \leqslant N \\ 1 & \text{for} \quad n > N. \end{cases} \tag{8}$$

The brackets $\langle\cdot\rangle$ indicate the average over an n-interval small compared with the dimension N. Assuming that the above relation also holds for the semiclassical traces $\langle|\,\mathrm{Tr}\,\tilde{\boldsymbol{U}}^n(N)|^2\rangle$ we can insert (8) in (7) which leads to the asymptotic result (Argaman *et al* 1993)

$$P(s,n) = \overline{P}(n) - \left(\frac{\sin n\pi s}{\pi s}\right)^2 + n\delta(s) \qquad \text{for} \quad |s| \ll 1. \tag{9}$$

Here, $\overline{P}(n) = \sum_{p,p'}^{(n)} A_p A_{p'}^*$ is the mean part of the weighted periodic orbit action pair density. Note that the number of periodic orbits increase exponentially with n for chaotic maps which in turn implies an exponential increase for the density of periodic orbit actions (modulus 1) and in general also for the weighted density $\overline{P}(n)$.

After rescaling the s-variable according to $s = \sigma/n$, equation (9) can be written as

$$P_{scal}^{GUE}(\sigma) = \frac{1}{n^2}P\left(\frac{\sigma}{n}\right) = \frac{\overline{P}(n)}{n^2} - \left(\frac{\sin\pi\sigma}{\pi\sigma}\right)^2 + \delta(\sigma) \qquad \text{for} \quad |\sigma| \ll n \tag{10}$$

which puts the non-trivial, σ-dependent correlations into n-independent form.

The corresponding equation for systems with time-reversal symmetry is (Argaman *et al* 1993)

$$\begin{aligned} P_{scal}^{GOE}(\sigma) = {} & \frac{\overline{P}(n)}{n^2} - 2\left(\frac{\sin\pi\sigma}{\pi\sigma}\right)^2 \\ & + \frac{2}{\pi\sigma}[\cos 2\pi\sigma\,(\mathrm{si}(2\pi\sigma)\cos 2\pi\sigma - \mathrm{Ci}(2\pi\sigma)\sin 2\pi\sigma) \\ & + \mathrm{Ci}(4\pi\sigma)\sin 4\pi\sigma - \mathrm{si}(4\pi\sigma)\cos 4\pi\sigma] + 2\delta(\sigma) \qquad \text{for} \quad |\sigma| \ll n \end{aligned} \tag{11}$$

with $\mathrm{si}(x)$, $\mathrm{Ci}(x)$ being the sine- and cosine-integrals.

For integrable systems conjectured to have Poissonian spectral statistics one expects

$$\frac{1}{N}\langle|\,\mathrm{Tr}\,\boldsymbol{U}^n(N)|^2\rangle = \left\langle \sum_{p,p'}^{(n)} A_p A_{p'}^* \mathrm{e}^{2\pi \mathrm{i} N(S_p - S'_p)} \right\rangle = 1 \qquad \text{for} \quad |N| \neq 0$$

and thus

$$P^{Pois}(s,n) = \sum_{N=-\infty}^{\infty} \frac{1}{N}|\,\mathrm{Tr}\,\tilde{\boldsymbol{U}}^n(N)|^2 \mathrm{e}^{2\pi \mathrm{i} s N} = \overline{P}(n) - 1 + \delta(s) \tag{12}$$

i.e. periodic orbit actions are uncorrelated.

The δ-functions in (9)–(12) can be identified with the diagonal terms $p = p'$ in the sum (7) after making use of the classical sum rule (Hannay and Ozorio de Almeida 1984)

$$\sum_p^{(n)} |A_p|^2 \to 1 \qquad \text{for large } n. \tag{13}$$

Before investigating the existence of correlations of the form (10) or (11) in more detail, I would like to make a few remarks: the weighted periodic orbit correlation function (7) contains a background term $\overline{P}(n)$ which, in general, increases exponentially with n. This is in contrast to the linear behaviour of the mean quantum level density $\overline{d}(N)$ in (2) and (3). Note also, that the complex weights A_p carry phases which may result in cancellations in $\overline{P}$ and may occasionally lead to $\overline{P} = 0$. The correlation functions (10) and (11) are in addition *not* rescaled with respect to the mean action density as e.g. the two-point spectral correlation function R_2 in (3). The differences in action between adjacent orbits of the same length n is exponentially small on the s or σ scale and the term $-(\sin\pi\sigma/\pi\sigma)^2$ in (10), (11) indicates long-range correlations on the scale of the mean periodic action separation. The correlations (10) and (11) imposed by RMT are thus a small modulation of order $\mathcal{O}(1)$ on top of an exponentially large background when considering all periodic orbit pairs (Dahlqvist 1995). The two-point correlation function for periodic orbit actions approaches the Poissonian limit for $n \to \infty$ when rescaling $P(\sigma, n)$ with respect to the mean action density in accordance to e.g. the definition of the spectral correlation function (3); periodic orbits are thus uncorrelated on scales of the mean action separation in agreement with the numerical results by Harayama and Shudo (1992).

The search for an $\mathcal{O}(1)$–effect on top of an exponentially large background is one of the main obstacles when investigating RMT-induced classical action correlations numerically. The second major problem in studying periodic orbit properties especially in the large n-limit is the exponential increase in the number of orbits with increasing n. In the following I will focus on a specific example, the classical and quantum Baker map, in which the problems mentioned earlier can be overcome by constructing a quasiclassical Perron–Frobenius-type operator (Dittes *et al* 1994).

3. The Baker map

The Baker map has become a standard example when studying classical and quantum chaos (see e.g. Balazs and Voros 1989, Saraceno and Voros 1994, O'Connor *et al* 1992, Hannay *et al* 1994). The classical dynamics is given by a two-dimensional, piecewise linear, area-preserving map on the unit square defined as

$$\begin{aligned} q' &= 2q - \epsilon \\ p' &= \frac{1}{2}(p + \epsilon) \qquad \text{with} \quad \epsilon = [2q] \end{aligned} \tag{14}$$

and (q, p), (q', p') being the initial and final points. The notation $[x]$ stands for the integer part of x. The Baker map is hyperbolic with a well defined Markov partition and a complete binary symbolic dynamics given in terms of the $\epsilon = \{0, 1\}$. The map is invariant under the anti-unitary symmetry $T : (q, p) \to (p, q)$ (being equivalent to time-reversal symmetry) and a parity transformation $R : (q, p) \to (1 - q, 1 - p)$. A proper desymmetrization of quantum spectra and semiclassical periodic orbit sums with respect to parity will be crucial when studying spectral statistics and periodic orbit correlations in section 4.

Each periodic orbit of the map can be associated with a finite symbol string $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ and the phase space coordinates of the orbit are given by the expression

$$q_\epsilon = \frac{1}{2^n - 1}\sum_{i=0}^{n-1} \epsilon_i 2^{n-i-1} \qquad p_\epsilon = \frac{1}{2^n - 1}\sum_{i=0}^{n-1} \epsilon_i 2^i. \tag{15}$$

A suitable generating function of the map is

$$W_\epsilon(p', q) = 2p'q - \epsilon p' - \epsilon q \qquad \text{with} \quad \epsilon = [2q] \tag{16}$$

which in turn defines the action; the action of a periodic orbit $\boldsymbol{\epsilon}$ of the map can, up to an additive integer, be written as (Dittes *et al* 1994)

$$S_\epsilon = \sum_{i=1}^{n} (q_i - 1)\epsilon_i = \sum_{i=1}^{n} (q_i - 1)[2q_i] \tag{17}$$

with q_i being the q-coordinates of the periodic orbit. The sum in (17) is invariant under time-reversal symmetry and parity-transformation $R(q_i) = 1 - q_i$; the later can be shown using the relation $\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \epsilon_i$.

A quantized version of the Baker map is obtained by making use of a discretized version of the generating function (16) and the classical structure of the map in the mixed representation. A suitable formulation preserving all the classical symmetries is provided by the choice (Balazs and Voros 1989, Saraceno and Voros 1994)

$$\boldsymbol{U}(N) = \boldsymbol{F}_N^{-1} \times \begin{pmatrix} \boldsymbol{F}_{N/2} & 0 \\ 0 & \boldsymbol{F}_{N/2} \end{pmatrix} \tag{18}$$

with

$$(\boldsymbol{F}_N)_{\mathrm{ij}} = \frac{1}{\sqrt{N}} \mathrm{e}^{-2\pi \mathrm{i}(\mathrm{i}-\frac{1}{2})(\mathrm{j}-\frac{1}{2})/N} \qquad i, j = 1, \ldots N. \tag{19}$$

The Fourier-matrix $\boldsymbol{F}$ is nothing but the transformation from position to momentum representation in the finite dimensional Hilbert space. The dimension N of the map $\boldsymbol{U}(N)$ has to be even in this construction and N is equivalent to the inverse of Planck's constant. The unitary map $\boldsymbol{U}(N)$ commutes with the parity operator $(\boldsymbol{R}_N)_{\mathrm{i,j}} = \delta_{\mathrm{i},N-\mathrm{j}+1}$ due to the particular choice of half-integer phases (Saraceno and Voros 1994). Symmetry reduction is obtained by considering the matrices $\boldsymbol{U}_\pm(N)$ defined as

$$\boldsymbol{U}_\pm(N) = \tfrac{1}{2}(\boldsymbol{U}(N) \pm \boldsymbol{U}(N)\boldsymbol{R}_N) \tag{20}$$

acting on the symmetric/anti-symmetric wavevectors only. This leads effectively to a reduction in dimension by a factor of two. The traces of $\boldsymbol{U}_\pm^n$ can in the large N-limit be written as (Saraceno and Voros 1994, Toscano *et al* 1997)

$$\mathrm{Tr}\, \boldsymbol{U}_\pm^n(N) = \frac{1}{2} \sum_{\epsilon}^{(n)} \left(\frac{2^{n/2}}{2^n - 1} \mathrm{e}^{2\pi \mathrm{i} N S_\epsilon} \pm \frac{2^{n/2}}{2^n + 1} \mathrm{e}^{2\pi \mathrm{i} N S'_\epsilon + \mathrm{i}\pi} + a_\epsilon \log N + b_\epsilon \right) + \mathcal{O}(N^{-1/2}) + \cdots. \tag{21}$$

The first two terms in the sum over the possible finite symbol strings of length n are the usual semiclassical Gutzwiller-like periodic orbit contributions. They arise as stationary phase points in a continuum approximation of $\mathrm{Tr}\, \boldsymbol{U}^n$, $\mathrm{Tr}(\boldsymbol{U}^n \boldsymbol{R})$, respectively, i.e. sums over matrix elements are replaced by integrals (Tabor 1983). The action S_ϵ of a periodic orbit of length n is given by equation (17). S'_ϵ corresponds to half the action of an orbit of length $2n$ with symbol code $(\epsilon_1, \ldots \epsilon_n, \bar{\epsilon}_1, \ldots \bar{\epsilon}_n)$ and $\bar{\epsilon}_i = 1 - \epsilon_i$. These are the orbits being invariant under the classical parity transformation R and are hence the stationary phase contributions of $\mathrm{Tr}(\boldsymbol{U}^n \boldsymbol{R})$. The anomalous $a_\epsilon \log N + b_\epsilon$ corrections are due to the discretization of the phase-space and arise in 'sub leading' integrals when performing Poisson summation on $\mathrm{Tr}\, \boldsymbol{U}^n$ i.e. replacing sums by sums over integrals (Saraceno and Voros 1994, Toscano *et al* 1997)). Next leading terms are diffraction corrections arising from the discontinuous nature of the classical and quantum map (14) and (18). Note the extra phase π in the $\mathrm{Tr}(\boldsymbol{U}^n \boldsymbol{R})$ contributions which originates from the discretized stationary phase approximation.

In the following I will mainly concentrate on the Gutzwiller-like periodic orbit contributions, i.e. I will consider the approximation

$$\begin{aligned} \mathrm{Tr}\, \boldsymbol{U}^n(N) &\approx \mathrm{Tr}\, \tilde{\boldsymbol{U}}^n(N) = \frac{2^{n/2}}{2^n - 1} \sum_{\epsilon}^{(n)} \mathrm{e}^{2\pi \mathrm{i} N S_\epsilon} \\ \mathrm{Tr}\, \boldsymbol{U}_\pm^n(N) &\approx \mathrm{Tr}\, \tilde{\boldsymbol{U}}_\pm^n(N) = \frac{1}{2} \left[\frac{2^{n/2}}{2^n - 1} \sum_{\epsilon}^{(n)} \mathrm{e}^{2\pi \mathrm{i} N S_\epsilon} \mp \frac{2^{n/2}}{2^n + 1} \sum_{\epsilon}^{(n)} \mathrm{e}^{2\pi \mathrm{i} N S'_\epsilon} \right]. \end{aligned} \tag{22}$$

I will henceforth refer to the periodic orbit sums (22) as the semiclassical approximation of the quantum Baker map. The influence of the remaining contributions in (21), including the leading order ($\log N$)-corrections, will be discussed at the end of this section.

The main problem in calculating periodic orbit sums like (22) is the exponential increase of periodic orbit contributions. Even for the Baker map where periodic orbit actions are given by the simple analytic formula (17) a direct summation of $\mathrm{Tr}\,\tilde{\boldsymbol{U}}^n$ becomes a computational challenge for $n \geqslant 30$. For the Baker map it is, however, possible, to construct a quasiclassical operator $\tilde{U}$ whose traces coincide with the semiclassical periodic orbit sum (22) (Dittes *et al* 1994). The spectrum of this operator can be computed explicitly and simplifies the task of calculating traces, i.e. periodic orbit sums, considerably. Other quasiclassical operators for the Baker map have been proposed by Kaplan and Heller (1996), Sano (1999) and Hannay (1999). Note that an identification of semiclassical periodic orbit sums with classical or quasiclassical operator is not possible for generic systems and the notation $\mathrm{Tr}\,\tilde{\boldsymbol{U}}^n$ is then a mere substitution for the periodic orbit sum itself.

The Baker map is special for the following reasons: firstly, the periodic orbit amplitudes A_p, (cf equation (6)), do not depend on the specific orbit, but only on the topological length n, which is a consequence of the piecewise linearity of the classical map. Secondly, the dynamics of the q-coordinate as well as the periodic action-orbit (17) are independent of the momentum p which allow for a classical separation of momentum and configuration space variables. As a consequence, one can define a one-dimensional Perron–Frobenius-type integral kernel acting on the q-coordinate only (Dittes *et al* 1994). In a form preserving the parity symmetry $R(q) = 1 - q$ it can be written as

$$\tilde{U}(q, q'; N) = \sqrt{2}\delta(q - (2q' - [2q']))\mathrm{e}^{2\pi \mathrm{i} N S(q')} \tag{23}$$

with

$$S(q') = q'[2q'] - \tfrac{1}{2}([2q'] + q'). \tag{24}$$

One easily deduces $\tilde{U}(q, q') = \tilde{U}(Rq, Rq')$. The traces of $\tilde{U}(q, q')$ coincide with the periodic orbit sum (22) if the operator is defined on the space of analytic functions on the unit interval with periodic boundary conditions (Dittes *et al* 1994, Rugh 1992). The operator is thus infinite-dimensional (having infinitely many eigenvalues) in contrast to the finite-dimensional quantum matrix $\boldsymbol{U}(N)$ and N is a mere parameter in (23). A symmetry-reduced version of this quasiclassical operator is obtained by considering

$$\tilde{U}_{\pm}(q, q'; N) = \tfrac{1}{2}(\tilde{U}(q, q'; N) \mp \tilde{U}(q, Rq'; N)). \tag{25}$$

Note that the quasiclassical operator acting on antisymmetric functions (denoted $\tilde{U}_+$ here) corresponds to the symmetric quantum subspace $\boldsymbol{U}_+$ and vice versa, reflecting the 'quantum' origin of the extra phase for the periodic orbit contributions to $\mathrm{Tr}\,\boldsymbol{U}^n\boldsymbol{R}$, see equation (21). The operator (23) or (25) can now be represented as an infinite-dimensional matrix after choosing a suitable basis set and the calculation of traces can be reduces to matrix calculus (provided the operator is trace class). An obvious choice for the basis functions is the Fourier basis, i.e.

$$\begin{aligned}\tilde{U}_{k,m} &= \int_0^1 \mathrm{d}q \int_0^1 \mathrm{d}q'\, \tilde{U}(q, q'; N)\mathrm{e}^{2\pi\mathrm{i}(mq' - kq)} \\ &= (-1)^m \sqrt{2}\frac{\mathrm{e}^{-\pi\mathrm{i}N/2}}{2\pi i}\left[\frac{\mathrm{e}^{\pi\mathrm{i}(N/2+m-2k)} - 1}{N/2 + m - 2k} + \frac{\mathrm{e}^{\pi\mathrm{i}(N/2-m+2k)} - 1}{N/2 - m + 2k}\right].\end{aligned} \tag{26}$$

Similarly one obtains

$$(\tilde{U}R)_{k,m} = \int_0^1 \mathrm{d}q \int_0^1 \mathrm{d}q'\, \tilde{U}(q, Rq'; N)\mathrm{e}^{2\pi\mathrm{i}(mq' - kq)}$$

$$= (-1)^m \sqrt{2} \frac{\mathrm{e}^{-\pi \mathrm{i} N/2}}{2\pi \mathrm{i}} \left[\frac{\mathrm{e}^{\pi \mathrm{i}(N/2+m+2k)} - 1}{N/2 + m + 2k} + \frac{\mathrm{e}^{\pi \mathrm{i}(N/2-m-2k)} - 1}{N/2 - m - 2k} \right] \quad (27)$$

with k, m integers. The various terms $(\mathrm{e}^{\pi \mathrm{i}(N/2 \pm m \pm 2k)} - 1)/(N/2 \pm m \pm 2k)$ are equal to $\mathrm{i}\pi$ for $N/2 \pm m \pm 2k = 0$.

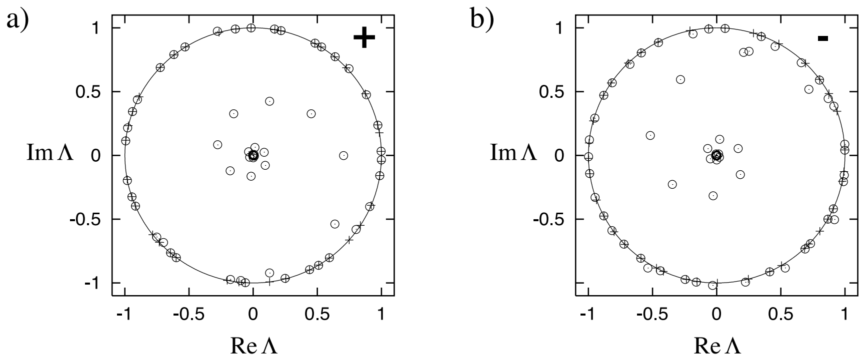

Figure 1. Semiclassical (○) and quantum (+) eigenvalues of the Baker map for $N = 80$; the parity subspaces are shown separately: (*a*) positive parity, (*b*) negative parity.

A typical eigenvalue spectrum $\{\Lambda_i\}$, $i = 0, 1, 2 \ldots$ of the quasiclassical operator $\tilde{U}_\pm$ is shown in figure 1 together with the quantum eigenvalues. In the following, the eigenvalues will be ordered with decreasing modulus, i.e. $|\Lambda_i| > |\Lambda_j|$ for $i < j$. The operator $\tilde{U}$ approximates unitarity of the corresponding quantum map $\boldsymbol{U}$ quite well; approximately $N/2$ eigenvalues in each parity subspace lie near the unit circle and these eigenvalues agree well with quantum results. The modulus of semiclassical eigenvalues becomes exponentially small for $i > N/2$ in each subspace and infinitely many eigenvalues 'disappear' by spiraling into the origin. This behaviour reflects the trace-class property of the operator and makes it possible to truncate the size of the matrix $\tilde{\boldsymbol{U}}_\pm$ in actual calculations. (Choosing $\dim(\tilde{\boldsymbol{U}}_\pm) = 3 \times \frac{N}{2}$ ensures convergence of the first $N/2$ eigenvalues in each parity subspace to four significant digits.)

A measure for the error of the semiclassical approximation compared with quantum calculations is the deviation of the semiclassical eigenvalue with largest modulus, $\Lambda_0(N)$, from the unit circle. The distribution of $\gamma^\pm(N) := \log|\Lambda_0^\pm(N)|$ for integer values of N and both subspaces is shown in figure 2. The deviation of semiclassical eigenvalues from the unit circle decreases with increasing N following a $1/\sqrt{N}$ behaviour. The $1/\sqrt{N}$-scaling suggest that corrections to individual semiclassical eigenvalues are dominated by diffraction contributions. The $\log(N)$ corrections in the traces, cf (21), become dominant when summing over the periodic orbit contributions or equivalently over the semiclassical eigenvalues and are thus a collective effect of the various contributions.

The deviations from the unit circle for semiclassical eigenvalues with modulus greater than one is systematically larger for eigenvalues corresponding to the $\tilde{U}_-$-operator compared with those from the $\tilde{U}_+$-operator. This can qualitatively be understood for small N when considering the limit $N = 0$; the operator $\tilde{U}(q, q'; N = 0) = \sqrt{2}\delta(q - (2q' - [2q']))$ is the classical Perron–Frobenius operator for the Sawtooth map up to a factor of $\sqrt{2}$. The spectrum $\tilde{U}(q, q'; 0)$ has therefore a single non-zero eigenvalue $\sqrt{2}$ in the symmetric subspace which corresponds to U_- in our notation; all other eigenvalues are zero. The eigenvalues are smooth functions of the continuous variable N which leads to $|\Lambda_0^-| > |\Lambda_0^+|$ for the N smaller one. The fact that this behaviour persists for large N values, cf figure 1, can certainly not be explained by the argument above but is worth noting as a purely numeric result.

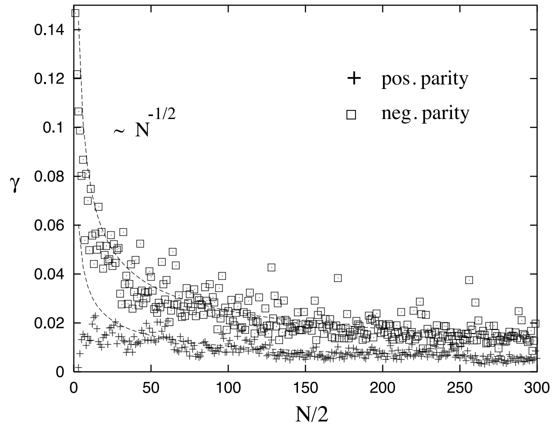

Figure 2. The semiclassical eigenvalue with the largest modulus plotted here as $\gamma^{\pm}(N) = \log|\Lambda_0^{\pm}(N)|$ versus $N/2$ for both subspaces; the deviation from the unit circle decreases in both cases like $1/\sqrt{N}$ (dashed curves), but one finds typically $\gamma^{+}(N) < \gamma^{-}(N)$.

4. Semiclassical spectral statistics and action correlation functions: numerical results

The existence of a quasiclassical operator for the Baker map makes it possible to write traces as convergent sums over the quasiclassical eigenvalue spectrum, i.e.

$$\operatorname{Tr} \tilde{U}_{\pm}^{n}(N) = \sum_{i=0}^{\infty} (\Lambda_i^{\pm}(N))^n. \tag{28}$$

The violation of unitarity and the existence of semiclassical eigenvalues with modulus greater than one leads to the asymptotic result

$$|\operatorname{Tr} \tilde{U}_{\pm}^{n}(N)| \approx \exp(n\gamma^{\pm}(N)) \qquad \text{for} \quad n \to \infty$$

with $\gamma^{\pm}(N) = \log|\Lambda_0^{\pm}(N)|$, i.e. the traces grow exponentially in the limit $n \to \infty$ (Keating 1994, Aurich and Sieber 1994). This kind of behaviour is connected to the notorious convergence problems for periodic orbit sums of the form

$$\sum_{n=1}^{\infty} \sum_{p}^{(n)} A_p \mathrm{e}^{2\pi \mathrm{i} N S_p}$$

which are absolutely convergent only for complex N with $\operatorname{Im}(N) > h_t/2$ and h_t is the topological entropy (Eckhard and Aurell 1989). For the Baker map the absolute sum over periodic orbit contributions (6) corresponds to $\sum_{n=1}^{\infty} \operatorname{Tr} \tilde{U}^n(0)$; this sum diverges like $2^{n/2}$ for large n, i.e. $h_t = \log 2$ here. A summation over the semiclassical eigenvalues (28) gives a detailed account of the regime of conditional convergence for periodic orbit sums as a function of the real part of N.

The exponential increase in the traces (28) leads to exponentially increasing terms in a semiclassical approximation of the form factor

$$K_{sc}(\tau, N) = \frac{1}{N}|\operatorname{Tr} \tilde{U}_{\pm}^{N\tau}|^2 = \frac{1}{N} \sum_{i,j=0}^{\infty} (\Lambda_i^{\pm} \Lambda_j^{\pm *})^{N\tau} \approx \exp(2N\tau\gamma^{\pm}(N)) \qquad \text{for} \quad \tau \to \infty. \tag{29}$$

A stationary distribution of K_{sc} for fixed τ and $N \to \infty$ is obtained if $\langle\gamma(N)\rangle$ falls off faster than N^{-1}. Such a behaviour is expected for smooth hyperbolic maps where the semiclassical error is dominated by $1/N$-corrections due to higher order terms in a stationary phase expansion. This is, however, not the case for the Baker map where diffraction effects of the order $N^{-1/2}$ dominate, see figure 2.

Using the semiclassical traces Tr $\tilde{U}^n$ to evaluate the two-point correlation function (4) will, on the other hand, always be affected by the exponentially increasing terms in the form factor; a semiclassical calculation of $R_2(x, N)$ thus diverges independently of N and the behaviour of $\langle\gamma(N)\rangle$.

The weighted classical correlation function (7) is a finite sum and is thus well defined for fixed topological length n. From (22) one obtains in the large n-limit

$$\overline{P}(n) = 2^{-n} \qquad \text{in the '+' - subspace}$$
$$\overline{P}(n) = 2^{n} \qquad \text{in the '$-$' - subspace}$$

and we will discard this 'trivial' mean part from now on. The large n-asymptotics of the non-trivial correlations in (7) is dominated by the eigenvalue being the largest among all $\Lambda_0(N)$; after defining

$$\gamma_0^{\pm} := \log|\Lambda_0^{\pm}(N_0)| := \sup_{N\in\mathbb{N}}(\log|\Lambda_0^{\pm}(N)|)$$

where N_0 is the N-value corresponding to the largest eigenvalue Λ_0, one obtains

$$P(s,n) \to \mathrm{e}^{2n\gamma_0^{\pm}}\cos(2\pi N_0 s) \to \infty \qquad \text{for} \quad n \to \infty. \tag{30}$$

It follows from the considerations above that a full description of quantum spectral statistics in terms of classical actions and amplitudes needs to incorporate unitarity into a semiclassical approximation (Keating 1994). I will come back to this point at the end of the section. Apart from that there seems to be little information in the asymptotic behaviour of $K_{sc}(\tau, N)$ or $P(s, n)$. The rates $|\gamma^{\pm}(N)| > 0$ correspond to 'semiclassical escape rates' and contain little information about the classical dynamics or the quantum map. The distribution of these rates is instead of statistical nature due to accumulation of errors in the semiclassical approximation.

The more interesting question in this context is whether or not semiclassical periodic orbit formulae for classical chaotic systems are able to reproduce RMT statistics quantitatively in the non-asymptotic regime. The form factor obtained from the quasiclassical operator is displayed in figure 3 together with the result obtained from the quantum spectrum and the GOE prediction (applicable for systems with time-reversal symmetry). The semiclassical result is indeed capable of reproducing the RMT result in both subspaces for small τ values; especially the deviation of the RMT behaviour from the classical result $K(\tau) = 2\tau$ obtained from the sum rule (13) in the limit $\tau \to 0$ is here well reproduced by periodic orbit formulae. This clearly indicates that there are non-trivial correlations between periodic orbit actions for systems with time-reversal symmetry.

Representing periodic orbit formulae by sums over quasiclassical eigenvalues allows one to study the large N and τ limit of a semiclassical approximation of the form factor. Exponentially increasing components start to dominate $K_{sc}(\tau, N)$ for τ larger than a semiclassical break-time of the order

$$\tau_b^{\pm} \sim \frac{1}{\gamma^{\pm}(N)N}. \tag{31}$$

The break-time $\tau_b^{\pm}$ is typically a factor 5 larger in the '+' subspace than in the '$-$' subspace, see figure 2.

The action correlation function (7), on the other hand, can be obtained either by sampling action-differences directly (which is limited to $n < 15$–20 due to the exponential increase in the number of orbits) or with the help of the traces Tr $\tilde{U}^n(N)$. The latter method demands

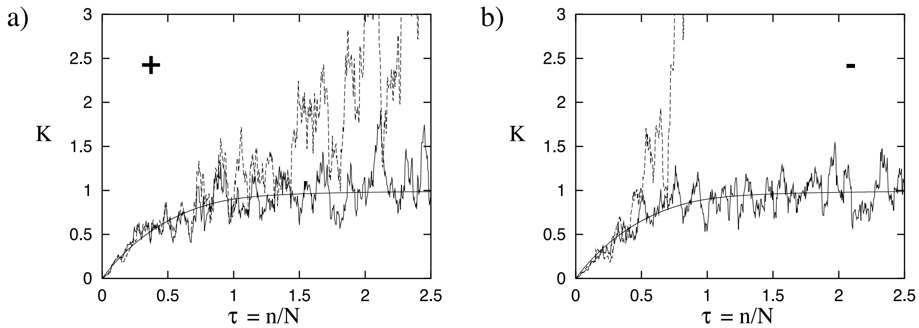

Figure 3. The form factor $K(\tau)$, i.e. the square modulus of the quantum traces $\mathrm{Tr}\, \boldsymbol{U}^n$ (full line) and the semiclassical traces $\mathrm{Tr}\, \tilde{U}^n$ (dashed line) for positive (*a*) and negative (*b*) parity is displayed as function of $\tau = n/N$ for $N/2 = 259$ together with the RMT result for the Gaussian orthogonal ensemble (smooth curve). The semiclassical break-times, equation (31), are $\tau_b^+ \approx 1.002$, $\tau_b^- \approx 0.301$, respectively.

calculation of the spectra of $\tilde{U}(N)$ for integer N-values and summation of the Fourier-sum in (7) directly. In practice, the sum is truncated at a finite N-value, $N_{\max}$, which corresponds to a smoothing of the original correlation function $P(s, n)$ on scales $1/N_{\max}$; after rescaling according to equation (10) and subtracting $\overline{P}(n)$ one obtains

$$P_{scal}(\sigma, n, N_{\max}) = \frac{2}{n^2} \sum_{N=1}^{N_{\max}} |\,\mathrm{Tr}\, \tilde{\boldsymbol{U}}^n(N)|^2 \cos\left(2\pi \frac{N}{n}\sigma\right) \approx \int_{-\infty}^{\infty} \mathrm{d}\sigma'\, P_{scal}(\sigma', n) g_\alpha(\sigma - \sigma') \tag{32}$$

with

$$g_\alpha(x) = \frac{\sin(2\pi\alpha x)}{\pi x} \qquad \text{and} \qquad \alpha = \frac{N_{\max}}{n}. \tag{33}$$

The smoothed periodic orbit correlation function $P_{scal}(\sigma, n, N_{\max})$ is shown in figure 4 for various n-values and $N_{\max} = n$. (Contributions from diagonal terms leading to the δ-functions in (9) and (10) are subtracted here.) Universal periodic orbit correlations in the '+' subspace are observed up to $n \approx 70$; these are 2^{70} distinct periodic orbits, a number inaccessible to pure periodic orbit calculations. One can study even longer orbits and finds exponentially increasing terms dominating the correlation function for n values above the transition point $n \approx 70$. The modulations in the periodic orbit pair density function $P(\sigma, n, n)$ are indeed orders of magnitudes larger than the RMT correlations for the 2^{500} periodic orbits of length $n = 500$, see figure 4(*c*).

Things look similar in the '−' subspace, the deviations from universality occur, however, for much smaller n-values, i.e. at $n \approx 6$. The correlations which occur for larger n have the simple form $P(s, n) = \mathrm{e}^{\gamma_0^- n} \cos(2\pi s)$, see figure 4(*d*) for $n = 65$. This oscillatory behaviour has already been observed by Sano (1999) when studying periodic orbit correlations in the Baker map without separating the two symmetry subspaces. Sano's result can now be interpreted in terms of the spectrum of the quasiclassical operator (23); the exponents $\gamma_0^\pm$ for the two different subspaces, which determine the asymptotic behaviour of $P(s, n)$ for large n, see equation (30), are

$$\gamma_0^+ = 0.022885 \qquad \text{with} \quad N_0 = 14$$
$$\gamma_0^- = 0.148508 \qquad \text{with} \quad N_0 = 1$$

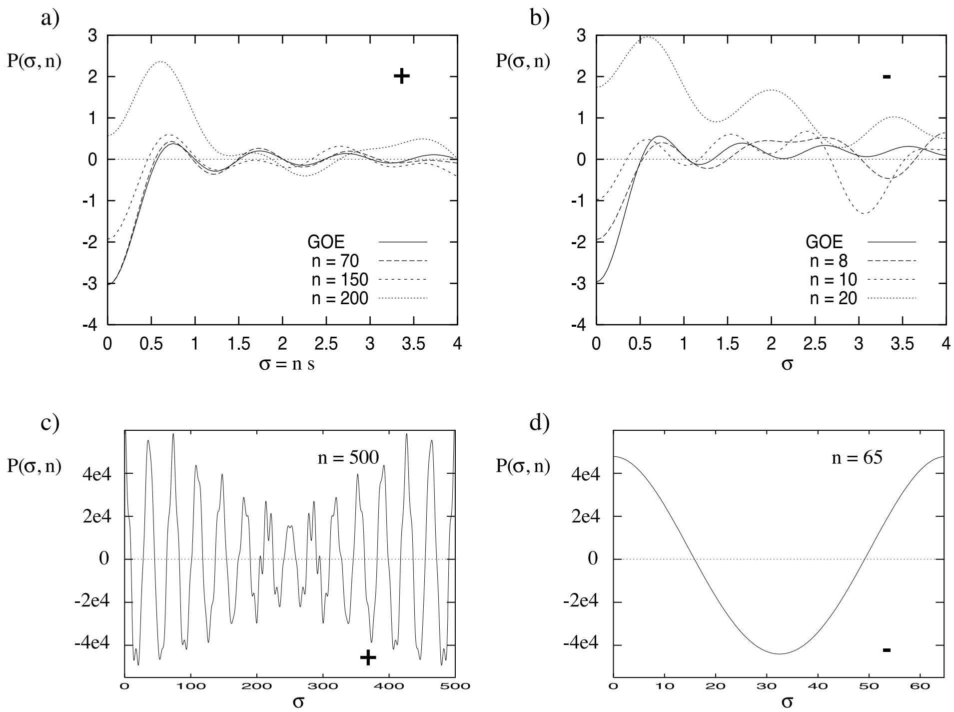

Figure 4. Periodic orbit correlation functions are shown for various n–values in the '+' subspace (a), (c) and '$-$' subspace (b), (d); 'universal' periodic orbit correlations which coincide with the GOE prediction can be observed up to $n \approx 70$ in the '+' subspace, but only up to $n \approx 8$ in the '$-$' subspace. Non-universal periodic orbit correlations develop for large n (with exponentially increasing amplitude), see (c) and (d). The cut-off is $N_{\max} = n$ here, see equation (32).

which can also be deduced from figure 2. The exponentially growing terms in the periodic orbit correlation function have thus a seven times larger leading exponent in the '$-$' subspace compared to the '+' subspace. The correlation function $P(s, n)$ in the '$-$' subspace is dominated by the Fourier coefficient $|\,\mathrm{Tr}\,\tilde{U}^n(1)|^2$ for $n \geqslant 6$ and RMT-type correlation cannot develop. The same is obviously true when considering the full Baker map without symmetry reduction.

Universal correlations coinciding with the RMT prediction do, however, exist. This is clearly demonstrated in the '+' subspace for $n \leqslant 70$. It is the violation of unitarity in a semiclassical approximation which leads (in general) to exponentially diverging terms which in turn overwhelm the pair-distribution function $P(s, n)$ for large n. The existence of universal periodic orbit correlations is thus strongly linked to the preservation of unitarity in a semiclassical formulation. The absence of RMT-like periodic orbit correlation, as observed in the '$-$' subspace, on the other hand, contains little information about the phenomena of universal spectral statistics but merely reflects the limitations of the approach due to the underlying semiclassical approximations.

A possibility to enforce unitarity onto a semiclassical description has been proposed by Bogomolny and Keating (1996b). The starting point is the quantum staircase function $\mathcal{N}(\theta) = \sum_{i=1}^{N} \Theta(\theta - \theta_i)$ with θ_i being the quantum eigenphases and $\Theta(x)$ denotes the Heaviside step-function. A smoothed, semiclassical version of the staircase function may be written as the truncated Fourier-series of $\mathcal{N}(\theta)$

$$\tilde{\mathcal{N}}(\theta, N) = \frac{N}{2\pi}\theta - \frac{1}{\pi}\mathrm{Im}\sum_{n=1}^{N}\frac{1}{n}\,\mathrm{Tr}\,\tilde{\boldsymbol{U}}^n(N)\mathrm{e}^{-\mathrm{i}n\theta} \tag{34}$$

and the traces $\mathrm{Tr}\,\tilde{\boldsymbol{U}}^n$ are expressed as sums over periodic orbit contributions, see equation (6). The unitarity condition is implemented by choosing a quantization condition

$$\tilde{\mathcal{N}}(\tilde{\theta}_n, N) \stackrel{!}{=} n + \tfrac{1}{2} \tag{35}$$

with $\tilde{\theta}_n$ being the solutions of (35); the $\tilde{\theta}_n$ represent a semiclassical approximation of the quantum eigenphases θ_n. A density of states is defined according to

$$D(\theta, N) = \sum_{i=1}^{N}\delta_{2\pi}(\theta - \tilde{\theta}_i) = \tilde{d}(\theta, N)\sum_{i=1}^{N}\delta_{2\pi}(\tilde{\mathcal{N}}(\theta, N) - n - \tfrac{1}{2}) \tag{36}$$

with $\tilde{d}(\theta, N) = \frac{\partial}{\partial\theta}\tilde{\mathcal{N}}(\theta, N)$. The density (36) is again written in terms of periodic orbit contributions only but 'preserves' unitarity by construction. Bogomolny and Keating (1996b) started from the expression (36) to derive periodic orbit corrections to the quantum two-point correlation function R_2 beyond the results obtained from the classical sum-rule (13).

The Perron–Frobenius representation of the semiclassical traces for the Baker map allows one to test the quantization condition (35) for large N up to $N \approx 500$ by calculating the semiclassical traces in (34) directly; the 'regularized' spectrum $\{\tilde{\theta}_i\}$ thus obtained is used to construct new semiclassical traces $\mathrm{Tr}\,\tilde{\tilde{U}}^n = \sum_i \exp(\mathrm{i}n\tilde{\theta}_i)$ which in turn are used to calculate the periodic orbit correlation function (7). Note, however, that the simple relation between periodic orbits of length n and the nth semiclassical trace gets lost here and the new periodic orbit correlation function $P(s, n)$ also contains correlation between orbits and pseudo-orbits (being composites of shorter orbits).

The correlation function obtained from the 'regularized' semiclassical data is shown in figure 5 together with correlation functions where regularization has not been applied, cf also figure 4. The functions $P(\sigma, n, N_{\mathrm{max}})$ are shown here with enhanced resolution compared with figure 4, i.e. with a cut-off $N_{\mathrm{max}} = \frac{5}{2}n$. The periodic orbit correlation function for $n = 70$ follows the random matrix prediction also with enhanced resolution, but again deviates from the universal behaviour for $n = 200$. The exponentially growing terms are eliminated in the corresponding regularized correlation function using the spectrum obtained from (35) for $n = 200$. The regularization procedure unveils the underlying universal correlations in the classical actions, see figure 5.

The regularization process (35) has the disadvantage that the Fourier components of the new density of states $D(\theta, N)$ can no longer be written as closed expressions of a finite number of periodic orbits; they become complicated infinite periodic orbit sums after writing the $\delta_{2\pi}$-function in its Fourier components (Bogomolny and Keating 1996b). The resulting periodic orbit correlation functions, as e.g. shown in figure 5 for $n = 200$, consist of periodic orbit contributions of orbits and pseudo-orbits of topological length up to and including $n = 200$ which does not simplify the task of understanding the origin of universal periodic orbit correlations.

5. Conclusions and outlook

Correlations between actions of periodic orbits of the Baker map up to orbits of topological length $n = 500$ (corresponding to 2^{500} different periodic orbits) have been studied with the help of a quasiclassical Perron–Frobenius operator. The spectral form factor can be calculated by

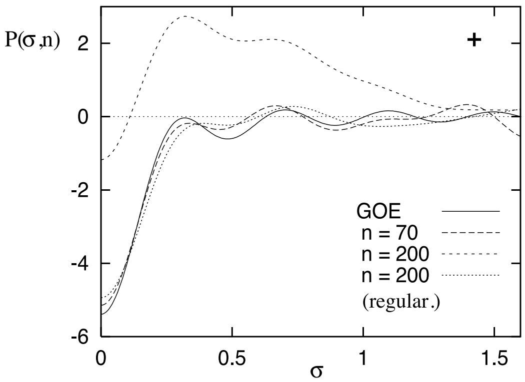

Figure 5. The periodic orbit correlation function shown for various n–values in the '+' subspace with enhanced resolution compared with figure 4; the regularization procedure (35) eliminates the exponentially growing terms and universal correlations are uncovered now also for $n = 200$.

purely semiclassical expressions which coincide for small τ with random matrix results beyond the validity of the classical sum-rule, but diverge for $\tau \to \infty$. Action correlations of periodic orbits have been investigated which show universal non-trivial correlations linked to random matrix theory for short periodic orbits, but depart from the universal behaviour for long orbits due to the violation of unitarity in a semiclassical approximation. The transition point from universal to non-universal statistics is distinctively different for the two parity subspaces in the Baker map. It is linked to the magnitude of the semiclassical error made when approximating the quantum density of states by semiclassical periodic orbit formulae and is controlled by the largest deviation of a semiclassical eigenvalue from the unit circle. This behaviour is probably generic for quantum maps. Statistical properties of semiclassical expressions are universal in the non-asymptotic regime, the transition point at which universality breaks down is, however, system dependent and controlled by the semiclassical error. Imposing unitarity on a semiclassical approximation makes it possible to discard exponentially growing terms in the periodic orbit correlation function which in turn uncovers universal correlation even above the transition point, as demonstrated here for the Baker map.

The study presented here unambiguously establishes for the first time the existence of universal periodic orbit correlations in a classical chaotic system whose quantum counterpart shows RMT-eigenvalue statistics; the limitations due to the semiclassical approximations are discussed in detail. This sets a proper framework for studying the *origin* of these classical correlations. A better understanding of the interplay between classical periodic orbit action correlations and unitarity might shed light on the existence of universality in quantum spectra in general.

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